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LOGARITHMIC FRACTIONAL SOBOLEV TRACE INEQUALITIES

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Abstract. Logarithmic fractional Sobolev trace inequalities are derived as a generalization of the results in [6, 9].

1. Introduction

The classical Sobolev trace inequalities are written as:

$$\left(\int_{\mathbb{R}^n} |f(x)|^q dx\right)^{1/q} \leq A_{p,q} \left(\int_{\mathbb{R}^{n+1}_+} |\nabla u(x,y)|^p dxdy\right)^{1/p} \tag{1.1}$$

with

$$\frac{1}{q} = \frac{n+1}{np} - \frac{1}{n},$$

where u is the extension of f to the upper half-space and $A_{p,q}$ is a positive constant independent of u. Many mathematicians have developed this type of inequalities using various methods in different settings (see, for example, [2, 3]). These inequalities of Sobolev type provide estimates of lower order derivatives of the trace function f in terms of higher order derivatives of u. Very recently this is generalized to nonhomogeneous fractional Sobolev spaces $W^{s,p}(\mathbb{R}^n)$ [6].

Logarithmic Sobolev trace inequalities capture the spirit of classical Sobolev trace inequalities with the logarithm function replacing powers, and they can be considered as limiting cases of Sobolev trace inequalities. It was first investigated in [8]: for $f \in \mathcal{S}(\mathbb{R}^n)$ with $\|f\|_{L^2(\mathbb{R}^n)} = 1$,

$$\int_{\mathbb{R}^n} |f(x)|^2 \ln |f(x)| dx \leq \frac{n}{2} \ln \left(A_n \int_{\mathbb{R}^{n+1}_+} |\nabla u(x,y)|^2 dxdy\right), \tag{1.2}$$

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where u is an extension of f to the upper half-space that is continuous in the closed upper half-space and at least once differentiable on the open upper half-space, and A_n is a positive constant dependent only on the dimension n. The logarithmic uncertainty principle was utilized to derive the logarithmic inequality (1.2).

A generalization of logarithmic Sobolev trace inequalities (1.2) was investigated in [9]: For any measurable function u satisfying $\nabla u \in L^p(\mathbb{R}^{n+1}_+)$ and

$$\|f\|_{L^q(\mathbb{R}^n)} = 1$$

with $u(x,0) = f(x)$ in the sense of distribution, we have

$$\left(\int_{\mathbb{R}^n} |f(x)|^q \ln |f(x)| dx\right) \leq \frac{n}{p} \ln \left(A_p \int_{\mathbb{R}^{n+1}_+} |\nabla u(x,y)|^p dx dy \right) \tag{1.3}$$

for some absolute constant A_p under the conjugate condition

$$\frac{n+1}{p} = \frac{n-1}{q} + 1.$$

This paper derives a logarithmic fractional Sobolev trace inequality which is a generalization of the logarithmic trace inequality (1.3) and is a limiting case of the fractional Sobolev trace inequalities discussed in [6].

2. The main theorem and notations

The fractional Sobolev spaces $W^{s,p}(\mathbb{R}^n)$ of functions with $s \in \mathbb{R}$ are defined as

$$W^{s,p}(\mathbb{R}^n) := \left\{ u \in \mathcal{S}'(\mathbb{R}^n) : \mathcal{F}_n^{-1}\left((1+|\xi|^2)^{s/2}\widehat{u}\right) \in L^p(\mathbb{R}^n) \right\},$$

where $\mathcal{S}'(\mathbb{R}^n)$ is the set of all tempered distributions on $\mathbb{R}^n$ and the Fourier transform $\widehat{u} = \mathcal{F}_n(u)$ on $\mathbb{R}^n$ of the function $u \in \mathcal{S}(\mathbb{R}^n)$ is defined by

$$\widehat{u}(\xi) = \mathcal{F}_n(u)(\xi) = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} u(x) e^{-ix\cdot\xi}\, dx.$$

The nonhomogeneous Sobolev space $W^{s,p}(\mathbb{R}^n)$ is equipped with the norm

$$\|u\|_{W^{s,p}} := \left(\int_{\mathbb{R}^n} \left| \mathcal{F}_n^{-1}\left((1+|\xi|^2)^{s/2}\widehat{u}\right)(x) \right|^p dx \right)^{1/p}.$$

Recently developed fractional Sobolev trace inequalities on $W^{s,q}(\mathbb{R}^n)$ are introduced:

PROPOSITION 1. [6] *Let p, r be extended real numbers of $\frac{1}{p}+\frac{1}{r}=1$, $1 \le p \le 2$ and let s be a real number satisfying*

$$s > (n+1)\left(\frac{1}{p}-\frac{1}{r}\right)+\frac{1}{r}. \tag{2.1}$$

Then for $u \in W^{s,p}(\mathbb{R}^{n+1})$ with the trace f on $\mathbb{R}^n$, we have

$$\|f\|_{L^r(\mathbb{R}^n)} \le C_{p,s}\|u\|_{W^{s,p}(\mathbb{R}^{n+1})}, \tag{2.2}$$

for some positive constant $C_{p,s}$.

The main theorem of this paper can be stated as follows:

THEOREM 1. *Let n be an integer with $n > 1$. Let p, q be real numbers satisfying $1 \le p \le 2$, $q \ge 1$ and*

$$\frac{1}{p}+\frac{n-1}{nq}=1.$$

For any u in $W^{s,p}(\mathbb{R}^{n+1})$ with $u(x,0)=f(x)$ in the sense of distribution and

$$\|f\|_{L^q(\mathbb{R}^n)}=1,$$

we have

$$\int_{\mathbb{R}^n}|f(x)|^q \ln|f(x)|dx \le n\ln\left(C_{p,s}\|u\|_{W^{s,p}(\mathbb{R}^{n+1})}\right) \tag{2.3}$$

under the condition

$$s > \frac{n+1}{p}-\frac{n-1}{q}=\frac{n(2-p)+1}{p}.$$

The constant $C_{p,s}$ is the same positive constant independent of u appeared in (2.2).

We point out that the inequality (2.3) with $p=2$ is reduced to the inequality (1.2).

3. The proof of the main theorem

We assume that f belongs to the Schwartz class $\mathcal{S}(\mathbb{R}^n)$ of functions on $\mathbb{R}^n$. We first state the following useful lemma.

LEMMA 2. [5] *Assume $f \in L^{p_0}(X,\mu)$ for some $0 < p_0 \le \infty$ and $\mu(X)=1$. Then we have $f \in L^p(X,\mu)$ for $0 < p \le p_0$, and that*

$$\lim_{p\to 0}\left(\int_X |f|^p d\mu\right)^{\frac{1}{p}} = \exp\left(\int_X \ln|f|d\mu\right).$$

From the assumption that $\|f\|_{L^q(\mathbb{R}^n)} = 1$, Lemma 2 with respect to the probability measure $|f(x)|^q dx$ yields

$$\begin{aligned}\lim_{r\to q}\left(\int_{\mathbb{R}^n}|f(x)|^r dx\right)^{\frac{1}{r-q}} &= \lim_{r\to q}\left(\int_{\mathbb{R}^n}|f(x)|^{r-q}|f(x)|^q dx\right)^{1/(r-q)} \\ &= \exp\left(\int_{\mathbb{R}^n}|f(x)|^q \ln|f(x)| dx\right).\end{aligned}$$

On the other hand, we split the index r into two numbers, and apply Hölder's inequality to get:

$$\begin{aligned}\int_{\mathbb{R}^n}|f(x)|^r dx &= \int_{\mathbb{R}^n}|f(x)|^{n(r-q)+\{r-n(r-q)\}}dx \\ &\leq \left(\int_{\mathbb{R}^n}|f(x)|^{n(r-q)\alpha}dx\right)^{\frac{1}{\alpha}}\left(\int_{\mathbb{R}^n}|f(x)|^{\{r-n(r-q)\}\beta}dx\right)^{\frac{1}{\beta}},\end{aligned}$$

where $1 < \alpha < \infty$ and $\frac{1}{\alpha} + \frac{1}{\beta} = 1$. Here β can be chosen to satisfy that

$$\{r - n(r-q)\}\beta = q.$$

Then the assumption $\|f\|_{L^q(\mathbb{R}^n)} = 1$ can be used to simplify the inequality as follows:

$$\begin{aligned}\left(\int_{\mathbb{R}^n}|f(x)|^r dx\right)^{\frac{1}{r-q}} &\leq \left(\int_{\mathbb{R}^n}|f(x)|^{n(r-q)\alpha}dx\right)^{\frac{1}{\alpha(r-q)}} \\ &= \left(\int_{\mathbb{R}^n}|f(x)|^{\frac{nq}{n-1}}dx\right)^{\frac{n-1}{q}}.\end{aligned}$$

The fractional Sobolev trace inequality, Proposition 1, can be applied to achieve that for $f \in \mathcal{S}(\mathbb{R}^n)$

$$\left(\int_{\mathbb{R}^n}|f(x)|^r dx\right)^{\frac{1}{r-q}} \leq \|f\|^n_{L^{\frac{nq}{n-1}}(\mathbb{R}^n)} \leq \left(C_{p,s}\|u\|_{W^{s,p}(\mathbb{R}^{n+1})}\right)^n$$

where the indices p and q satisfy

$$s > \frac{n+1}{p} - \frac{n-1}{q}.$$

We now take the limit on both sides of the above inequality to get

$$\exp\left(\int_{\mathbb{R}^n}|f(x)|^q \ln|f(x)| dx\right) \leq \left(C_{p,s}\|u\|_{W^{s,p}(\mathbb{R}^{n+1})}\right)^n.$$

For $f \in \mathcal{S}(\mathbb{R}^n)$ with $\|f\|_{L^q(\mathbb{R}^n)} = 1$, we have the generalized logarithmic fractional Sobolev trace inequality of the form:

$$\int_{\mathbb{R}^n} |f(x)|^q \ln |f(x)| dx \leq n \ln \left(C_{p,s} \|u\|_{W^{s,p}(\mathbb{R}^{n+1})} \right).$$

The density argument completes the proof.

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